

Parahoric Restriction for $\mathrm{GSp}(4)$

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Parahoric restriction is the parahoric analogue of Jacquet's functor. Fix an arbitrary parahoric subgroup of the group $\mathrm{GSp}(4, F)$ of symplectic similitudes of genus two over a local number field F/\mathbb{Q}_p . We determine the parahoric restriction of the non-cuspidal irreducible smooth representations in terms of explicit character values.

1 Introduction

For a reductive connected group \mathbf{G} over a local number field F/\mathbb{Q}_p , with group of F -valued points $G = \mathbf{G}(F)$, fix a compact parahoric subgroup $\mathcal{P} \subseteq G$ with pro-unipotent radical \mathcal{P}^+ . The *parahoric restriction functor* between categories of admissible representations

$$\mathbf{r}_{\mathcal{P}} : \mathrm{Rep}(G) \rightarrow \mathrm{Rep}(\mathcal{P}/\mathcal{P}^+), \quad (\rho, V) \mapsto (\rho|_{\mathcal{P}}, V^{\mathcal{P}^+}).$$

is the parahoric analogue of Jacquet's functor of parabolic restriction. It assigns to an admissible representation (ρ, V) of G the action of the Levi quotient $\mathcal{P}/\mathcal{P}^+$ on the space of invariants under \mathcal{P}^+ . The functor is exact and factors over semisimplification, so it is sufficient to study irreducible admissible representations.

For the group $\mathbf{G} = \mathrm{GSp}(4)$ of symplectic similitudes of genus two, we determine the parahoric restriction of non-cuspidal irreducible admissible representations in terms of explicit character values for the finite Levi quotient $\mathcal{P}/\mathcal{P}^+$. This has applications in the theory of Siegel modular forms of genus two, invariant under principal congruence subgroups of squarefree level.

1.1 Main result

For a proper parabolic subgroup of $G = \mathrm{GSp}(4, F)$ fix a cuspidal irreducible admissible complex linear representation σ of its Levi quotient. Let ρ be a subquotient of the normalized parabolic induction of σ to G . Then ρ is non-cuspidal and every non-cuspidal irreducible admissible representation of G arises this way.

The conjugacy classes of parahoric subgroups in G are represented by the standard parahoric subgroups $\mathcal{B}, \mathcal{P}, \mathcal{Q}, \mathcal{K}, \mathcal{J}$ described below. We determine the parahoric restriction of ρ with respect to each of these standard parahoric subgroups. If σ has positive depth, then the parahoric restriction of ρ is zero, because the depths of σ and ρ coincide [5, 5.2].

Theorem 1.1. *Suppose σ has depth zero. i) The parahoric restriction of ρ with respect to the standard hyperspecial parahoric subgroup \mathcal{K} is given by Table 3.¹*

ii) The parahoric restriction of ρ with respect to the standard Iwahori \mathcal{B} , standard Siegel parahoric \mathcal{P} and standard Klingen parahoric \mathcal{Q} is given by Table 4.

iii) The parahoric restriction of ρ with respect to the standard paramodular group \mathcal{J} is given by Table 5.

The proof is in Section 3, the tables are in Section 4. By character theory, we can determine invariants for every subgroup of the Levi quotients. For the full Levi we obtain parahori-spherical vectors.²

Corollary 1.2. *The parahori-spherical irreducible admissible representations of G are exactly the subquotients of the unramified principal series $\mu_1 \times \mu_2 \rtimes \mu_0$ for unramified characters μ_1, μ_2, μ_0 of F^\times . The dimension of parahori-spherical vectors is given by Table 1.*

Proof. For non-cuspidal irreducible admissible representations, the dimension of parahori-spherical vectors equals the multiplicity of the trivial representation in the parahoric restriction. Cuspidal representations are never parahori-spherical [1, 4.7], [5, 6.11]. \square

2 Preliminaries

Fix a nonarchimedean local number field F with finite residue field $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$ of order q . The valuation character $\nu = |\cdot|$ of F^\times is normalized such that $|\varpi| = q^{-1}$ for a uniformizing element $\varpi \in \mathfrak{p}$.

2.1 The parahoric restriction functor

Let G be the group of F -rational points of a connected reductive linear algebraic group over a non-archimedean local number field F . Let $\mathcal{P} \subseteq G$ be a parahoric subgroup with Levi decomposition

$$1 \rightarrow \mathcal{P}^+ \rightarrow \mathcal{P} \rightarrow \underline{\mathcal{P}} \rightarrow 1. \quad (1)$$

¹Partial results in this case have been obtained before by Breeding [2].

²This coincides with previous work by Roberts and Schmidt [6, Table A.15].

Table 1: Dimension of parahori-spherical vectors for unramified characters μ_1, μ_2, μ_0, ξ .

type	ρ of $\mathrm{GSp}(4, F)$	$\dim \rho^{\mathcal{K}}$	$\dim \rho^{\mathcal{J}}$	$\dim \rho^{\mathcal{P}}$	$\dim \rho^{\mathcal{Q}}$	$\dim \rho^{\mathcal{B}}$
I	$\mu_1 \times \mu_2 \rtimes \mu_0$	1	2	4	4	8
IIa	$\mu_1 \mathrm{St} \rtimes \mu_0$	0	1	1	2	4
IIb	$\mu_1 \mathbf{1} \rtimes \mu_0$	1	1	3	2	4
IIIa	$\mu_1 \rtimes \mu_0 \mathrm{St}$	0	0	2	1	4
IIIb	$\mu_1 \rtimes \mu_0 \mathbf{1}$	1	2	2	3	4
IVa	$\mu_0 \mathrm{St}_{\mathrm{GSp}(4, F)}$	0	0	0	0	1
IVb	$L(\nu^2, \nu^{-1} \mu_0 \mathrm{St})$	0	0	2	1	3
IVc	$L(\nu^{3/2} \mathrm{St}, \nu^{-3/2} \mu_0)$	0	1	1	2	3
IVd	$\mu_0 \mathbf{1}_{\mathrm{GSp}(4, F)}$	1	1	1	1	1
Va	$\delta([\xi_u, \nu \xi_u], \nu^{-1/2} \mu_0)$	0	0	0	1	2
Vb	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \mu_0)$	0	1	1	1	2
Vc	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \xi_u \mu_0)$	0	1	1	1	2
Vd	$L(\nu \xi_u, \xi_u \rtimes \nu^{-1/2} \mu_0)$	1	0	2	1	2
VIa	$\tau(S, \nu^{-1/2} \mu_0)$	0	0	1	1	3
VIb	$\tau(T, \nu^{-1/2} \mu_0)$	0	0	1	0	1
VIc	$L(\nu^{1/2} \mathrm{St}, \nu^{-1/2} \mu_0)$	0	1	0	1	1
VIId	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2} \mu_0)$	1	1	2	2	3

where \mathcal{P} is a finite reductive group over the residue field. For an admissible complex representation $\pi : G \rightarrow \mathrm{Aut}(V)$ on a complex vector space V , the action of \mathcal{P} preserves the subspace $V^{\mathcal{P}^+}$ of \mathcal{P}^+ -invariants in V . This defines a representation $(\rho|_{\mathcal{P}}, V^{\mathcal{P}^+})$ of $\mathcal{P}/\mathcal{P}^+ \cong \mathcal{P}$. An intertwining operator $V_1 \rightarrow V_2$ between admissible representations (ρ_1, V_1) and (ρ_2, V_2) of G defines a canonical \mathcal{P} -intertwiner $V_1^{\mathcal{P}^+} \rightarrow V_2^{\mathcal{P}^+}$.

Definition 2.1. The *parahoric restriction functor* for \mathcal{P} is the exact functor between categories of admissible complex linear representations

$$\mathbf{r}_{\mathcal{P}} : \mathrm{Rep}(G) \rightarrow \mathrm{Rep}(\mathcal{P}), \quad \begin{cases} (\rho, V) \mapsto (\rho|_{\mathcal{P}}, V^{\mathcal{P}^+}), \\ (V_1 \rightarrow V_2) \mapsto (V_1^{\mathcal{P}^+} \rightarrow V_2^{\mathcal{P}^+}). \end{cases} \quad (2)$$

For parahoric subgroups $\mathcal{P}_2 \subseteq \mathcal{P}_1 \subseteq G$, parahoric restriction is *transitive*

$$\mathbf{r}_{\mathcal{P}_2}(\rho, V) \cong \mathbf{r}_{\mathcal{P}_2/\mathcal{P}_1^+} \circ \mathbf{r}_{\mathcal{P}_1}(\rho, V), \quad (3)$$

where $\mathbf{r}_{\mathcal{P}_2/\mathcal{P}_1^+} : \mathrm{Rep}(\mathcal{P}_1) \rightarrow \mathrm{Rep}(\mathcal{P}_2)$ is the parabolic restriction functor with respect to the parabolic subgroup $\mathcal{P}_2/\mathcal{P}_1^+ \subseteq \mathcal{P}_1/\mathcal{P}_1^+ \cong \mathcal{P}_1$ and Levi quotient \mathcal{P}_2 . Compare Vignéras [13, 4.1.3].

The *depth* of an irreducible admissible representation ρ of G is defined in the sense of Moy and Prasad [5]. By definition, an irreducible smooth representation of G has depth zero if and only if it admits non-zero parahoric restriction with respect to some parahoric subgroup.

Let P be a parabolic subgroup of G with Levi subgroup M and let $S \subseteq M$ be a maximal F -split torus of G . Fix an admissible irreducible representation σ of M and an irreducible subquotient ρ of its parabolic induction to G . Let \mathcal{P} be a parahoric subgroup attached to a point in the apartment of S .

Proposition 2.2. *If σ has non-zero parahoric restriction with respect to the parahoric subgroup $M \cap \mathcal{P}$ of M , then ρ has non-zero parahoric restriction with respect to \mathcal{P} .*

Proof. Replace P by an associate parabolic with the same Levi subgroup, so that there is a monomorphism $\rho \hookrightarrow \text{Ind}_P^G(\sigma)$ in $\text{Rep}(G)$ [5, 2.5]. By Frobenius reciprocity, there is an epimorphism $\mathbf{r}_P(\rho) \twoheadrightarrow \sigma$ in $\text{Rep}(M)$, where \mathbf{r}_P denotes Jacquet's functor from $\text{Rep}(G)$ to $\text{Rep}(M)$. By exactness of the functor of $\mathcal{P}^+ \cap M$ -invariants there is an epimorphism

$$(\mathbf{r}_P(\rho))^{M \cap \mathcal{P}^+} \twoheadrightarrow \sigma^{M \cap \mathcal{P}^+}.$$

Since $M \cap \mathcal{P}^+$ is the pro-unipotent radical of $M \cap \mathcal{P}$, the right hand side is non-zero by assumption and therefore $(\mathbf{r}_P(\rho))^{M \cap \mathcal{P}^+}$ is also non-zero. Since $\mathcal{P}^+ \subseteq G$ admits Iwahori decomposition with respect to P and M [5, 4.2], there is a surjection of vector spaces $\rho^{\mathcal{P}^+} \twoheadrightarrow (\mathbf{r}_P(\rho))^{M \cap \mathcal{P}^+}$ [5, 2.2]. Especially, $\rho^{\mathcal{P}^+}$ is non-zero. \square

2.2 Parahoric restriction for $\text{GL}(1)$ and $\text{GL}(2)$

For $G = \text{GL}(1, F)$, the irreducible admissible representations are the smooth characters $\mu : F^\times \rightarrow \mathbb{C}^\times$. The unique parahoric subgroup is \mathfrak{o}^\times with pro-unipotent radical $1 + \mathfrak{p}$. If μ is tamely ramified or unramified, its parahoric restriction $\mathbf{r}_{\mathfrak{o}^\times}(\mu)$ is the character $\tilde{\mu} : \mathfrak{o}^\times / (1 + \mathfrak{p}) \rightarrow \mathbb{C}$ such that

$$\mu(x) = \tilde{\mu}(x(1 + \mathfrak{p})) \quad \text{for} \quad x \in \mathfrak{o}^\times. \quad (4)$$

If μ is wildly ramified, its parahoric restriction is $\tilde{\mu} = 0$.

For $G = \text{GL}(2, F)$ fix the standard Borel B of upper triangular matrices and the maximal torus T of diagonal matrices. The conjugacy classes of parahoric subgroups are represented by the standard hyperspecial parahoric subgroup $\mathcal{K} = \mathcal{K}_G = \text{GL}(2, \mathfrak{o})$ with $\mathcal{K}_G^+ = \mathcal{K}_G \cap (I_2 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ 0 & \mathfrak{p} \end{pmatrix})$ and the standard Iwahori $\mathcal{B}_G = \mathcal{K}_G \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}$ with $\mathcal{B} = \mathcal{B}_G^+ = \mathcal{B}_G \cap (I_2 + \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ 0 & \mathfrak{p} \end{pmatrix})$. We identify $\mathcal{K}_G / \mathcal{K}_G^+ \cong \text{GL}(2, \mathfrak{o}/\mathfrak{p}) \cong \text{GL}(2, q)$ by the canonical isomorphism and identify $\mathcal{B}_G / \mathcal{B}_G^+ \cong (\mathfrak{o}/\mathfrak{p})^\times \times (\mathfrak{o}/\mathfrak{p})^\times$ via $x \mapsto (x_{11}\mathfrak{p}, x_{22}\mathfrak{p})$.

The irreducible smooth representations of G are

1. the principal series $\mu_1 \times \mu_2 = \text{Ind}_B^G(\mu_1 \boxtimes \mu_2)$ with $\mu_1 \mu_2^{-1} \neq \nu^{\pm 1}$,
2. one-dimensional representations $\mu_1 \mathbf{1}_G = \mu_1 \circ \det$,
3. twists of the Steinberg representation $\mu_1 \text{St}_G = (\mu_1 \circ \det) \otimes \text{St}_G$,
4. cuspidal irreducible representations π

Table 2: Parahoric restriction for smooth irreducible representations of $\mathrm{GL}(2, F)$.

ρ of $\mathrm{GL}(2, F)$	$\mathbf{r}_{\mathcal{K}}(\rho)$ of $\mathrm{GL}(2, q)$	$\mathbf{r}_{\mathcal{B}}(\rho)$ of $\mathrm{GL}(1, q) \times \mathrm{GL}(1, q)$
$\mu_1 \times \mu_2$	$\widetilde{\mu_1} \times \widetilde{\mu_2}$	$\widetilde{\mu_1} \boxtimes \widetilde{\mu_2} + \widetilde{\mu_2} \boxtimes \widetilde{\mu_1}$
$\mu_1 \cdot \mathbf{1}_{\mathrm{GL}(2, F)}$	$\widetilde{\mu_1} \cdot \mathbf{1}_{\mathrm{GL}(2, q)}$	$\widetilde{\mu_1} \boxtimes \widetilde{\mu_1}$
$\mu_1 \cdot \mathrm{St}_{\mathrm{GL}(2, F)}$	$\widetilde{\mu_1} \cdot \mathrm{St}_{\mathrm{GL}(2, q)}$	$\widetilde{\mu_1} \boxtimes \widetilde{\mu_1}$
π depth zero	cuspidal irreducible	0
π positive depth	0	0

for smooth characters μ_1, μ_2 of F^\times .

Lemma 2.3. *The parahoric restriction of the irreducible admissible representations ρ of G at \mathcal{K} and \mathcal{B} is given by Table 2.³*

Proof. For a pair of smooth characters μ_1, μ_2 of F^\times , the parahoric restriction of $\mu_1 \times \mu_2$ at \mathcal{K}_G is

$$\mathbf{r}_{\mathcal{K}_G}(\mu_1 \times \mu_2) \cong \widetilde{\mu_1} \times \widetilde{\mu_2},$$

by a standard argument using Iwasawa decomposition $G = B\mathcal{K}_G$, compare Prop. 3.3. By exactness of parahoric restriction and Maschke's theorem, the exact sequence

$$0 \longrightarrow \mu_1 \mathbf{1}_{\mathrm{GL}(2, F)} \longrightarrow \nu^{-1/2} \mu_1 \times \nu^{1/2} \mu_1 \longrightarrow \mu_1 \mathrm{St}_{\mathrm{GL}(2, F)} \longrightarrow 0.$$

yields an isomorphism

$$\widetilde{\mu_1} \mathrm{St}_{\mathrm{GL}(2, q)} \oplus \widetilde{\mu_1} \mathbf{1}_{\mathrm{GL}(2, q)} \cong \widetilde{\mu_1} \times \widetilde{\mu_1} \cong \mathbf{r}_{\mathcal{K}_G}(\mu_1 \mathbf{1}_G) \oplus \mathbf{r}_{\mathcal{K}_G}(\mu_1 \mathrm{St}_G).$$

If μ_1 is tamely ramified or unramified, we have $\mathbf{r}_{\mathcal{K}_G}(\mu_1 \circ \det) = \widetilde{\mu_1} \circ \det$ because $\det(\mathcal{K}_G^+) = 1 + \mathfrak{p}$. Cuspidal irreducible representations of depth zero are compactly induced from the normalizer of \mathcal{K} [5, 6.8], the result is then implied by a theorem of Vignéras [12, Cor. 5.3]. For Iwahori restriction use transitivity (3). \square

2.3 The group $\mathrm{GSp}(4)$

The group $\mathbf{G} = \mathrm{GSp}(4)$ of symplectic similitudes of genus two is defined over \mathbb{Z} by the equation

$$J = \nu g J g^t \quad \text{for} \quad g \in \mathrm{GL}(4), \nu \in \mathrm{GL}(1) \quad \text{and} \quad J = \begin{pmatrix} & & & I_2 \\ & & -I_2 & \\ & I_2 & & \\ & & & \end{pmatrix}.$$

The similitude factor $\mathrm{sim}(g) = \nu$ is uniquely determined by g and defines a character $\mathrm{sim} : \mathbf{G} \rightarrow \mathrm{GL}(1)$. We fix the split torus \mathbf{T} of diagonal matrices and the standard parabolic subgroups

$$\mathbf{B} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \cap \mathbf{G}, \quad \mathbf{P} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \cap \mathbf{G}, \quad \mathbf{Q} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \cap \mathbf{G}.$$

³This is well-known, compare Vignéras [11, III.3.14] or Bushnell and Henniart [3, §14, §15].

The corresponding groups of F -rational points are G, T, B, P, Q .

For smooth characters μ_i of F^\times , $i = 0, 1, 2$, normalized parabolic induction of the character $T \rightarrow \mathbb{C}$, $\text{diag}(t_1, t_2, t_0/t_1, t_0/t_2) \mapsto \mu_1(t_1)\mu_2(t_2)\mu_0(t_0)$ via the standard Borel B yields the admissible representation $\mu_1 \times \mu_2 \rtimes \mu_0$ of G . For parabolic induction via P and Q the notation is analogous, compare Tadić [10].

A non-trivial additive character $\psi : F \rightarrow \mathbb{C}^\times$ gives rise to a generic character of U_B via $\psi_U : U_B \rightarrow \mathbb{C}$, $u \mapsto \psi(u_{12} + u_{24})$. An admissible representation ρ of G is *generic* if it admits a non-trivial U_B -intertwining operator $(\rho|_{U_B}, V) \rightarrow (\psi_U, \mathbb{C})$.

We review the classification of standard parahoric subgroups of $\text{GSp}(4, F)$. The character group $X^*(\mathbf{T}) = \text{Hom}(\mathbf{T}, \text{GL}(1))$ is generated as a free group by the characters $e_i : \text{diag}(t_1, t_2, t_0/t_1, t_0/t_2) \mapsto t_i$ for $i = 0, 1, 2$. The simple affine roots $\psi_0 = -(2e_1 - e_0) + 1$, $\psi_1 = e_1 - e_2$ and $\psi_2 = 2e_2 - e_0$ constitute the affine Dynkin diagram

$$\mathcal{C}_2 : \quad \begin{array}{c} \bigcirc \quad \rightleftarrows \quad \bigcirc \quad \rightleftarrows \quad \bigcirc \\ \psi_0 \quad \quad \psi_1 \quad \quad \psi_2 \end{array} .$$

Let $N(T)$ be the normalizer of T in G . The affine Weyl group $N(T)/\mathbf{T}(\mathfrak{o})$ is generated by the root reflections s_i at ψ_i for $i = 0, 1, 2$ and the Atkin-Lehner element u_1

$$s_0 = \begin{pmatrix} & & \varpi^{-1} \\ & 1 & \\ -\varpi & & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} & & 1 \\ & \varpi & \\ \varpi & & \end{pmatrix}.$$

The simple affine roots ψ_0 and ψ_2 are conjugate under u_1 . The closed standard alcove \mathcal{C} in the apartment attached to T is defined by $\psi_i(x) \geq 0$ for $i = 0, 1, 2$. To each facet in \mathcal{C} is attached one of the standard parahoric subgroups of $\text{GSp}(4, F)$:

1. the standard Iwahori subgroup \mathcal{B} , attached to \mathcal{C} ,

$$\mathcal{B} = \text{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathcal{B}^+ = \mathcal{B} \cap \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1 + \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & 1 + \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1 + \mathfrak{p} \end{pmatrix}$$

with Levi quotient $\mathcal{B}/\mathcal{B}^+ \cong \text{GL}(1, q)^3$ via $x \mapsto (x_{11}, x_{22}, \text{sim}(x))$,

2. the standard Siegel parahoric \mathcal{P} , attached to the facet $\psi_1^{-1}(0) \cap \mathcal{C}$,

$$\mathcal{P} = \text{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathcal{P}^+ = \mathcal{P} \cap \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1 + \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & 1 + \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}$$

with $\mathcal{P}/\mathcal{P}^+ \cong \text{GL}(2, q) \times \text{GL}(1, q)$ via $x \mapsto ((\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}), \text{sim}(x))$,

3. the standard Klingen parahoric \mathcal{Q} , attached to the facet $\psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathcal{Q} = \text{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathcal{Q}^+ = \mathcal{Q} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1+\mathfrak{p} \end{pmatrix}$$

with $\mathcal{Q}/\mathcal{Q}^+ \cong \text{GL}(1, q) \times \text{GSp}(2, q)$ via $x \mapsto (x_{11}, \begin{pmatrix} x_{22} & x_{24} \\ x_{42} & x_{44} \end{pmatrix})$,

4. the standard hyperspecial parahoric subgroup $\mathcal{K} = \text{GSp}(4, \mathfrak{o})$, attached to the facet $\psi_1^{-1}(0) \cap \psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathcal{K} = \text{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathcal{K}^+ = \mathcal{K} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

with the canonical map $\mathcal{K}/\mathcal{K}^+ \cong \text{GSp}(4, \mathfrak{o}/\mathfrak{p}) \cong \text{GSp}(4, q)$,

5. the standard paramodular subgroup \mathcal{J} with facet $\psi_0^{-1}(0) \cap \psi_2^{-1}(0) \cap \mathcal{C}$,

$$\mathcal{J} = \text{sim}^{-1}(\mathfrak{o}^\times) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{pmatrix}, \quad \mathcal{J}^+ = \mathcal{J} \cap \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & 1+\mathfrak{p} \end{pmatrix}$$

with $\mathcal{J}/\mathcal{J}^+ \cong (\text{GL}(2, q)^2)^0 := \{(a, b) \in \text{GL}(2, q)^2 \mid \det a = \det b\}$ via

$$x \mapsto \left(\begin{pmatrix} x_{11} & x_{13}\varpi \\ x_{31}\varpi^{-1} & x_{33} \end{pmatrix}, \begin{pmatrix} x_{22} & x_{24} \\ x_{42} & x_{44} \end{pmatrix} \right),$$

6. the parahoric $u_1^{-1}\mathcal{Q}u_1$ attached to the facet $\psi_0^{-1}(0) \cap \mathcal{C}$,

7. the hyperspecial parahoric $u_1^{-1}\mathcal{K}u_1$ attached to $\psi_0^{-1}(0) \cap \psi_1^{-1}(0) \cap \mathcal{C}$.

Conjugation by the Atkin-Lehner element u_1 preserves \mathcal{B} , \mathcal{P} and \mathcal{J} . The standard maximal parahorics are \mathcal{K} , \mathcal{J} and $u_1^{-1}\mathcal{K}u_1$.

There are double coset decompositions

$$\text{GSp}(4, F) = B\mathcal{K} = P\mathcal{J} = Q\mathcal{J} \sqcup Qs_1\mathcal{J} \quad (5)$$

for the standard parabolics B , P , Q , where \sqcup denotes the disjoint union. The proof is elementary and follows from Iwasawa and Bruhat decomposition.

For every parahoric subgroup of $\text{GSp}(4, F)$, the image of the pro-unipotent radical under the similitude character is $1 + \mathfrak{p}$. Therefore twisting a representation ρ by a tamely ramified or unramified character μ of F^\times commutes with parahoric restriction in the following sense:

$$\mathbf{r}_{\mathcal{K}}((\mu \circ \text{sim}) \otimes \rho) \cong (\tilde{\mu} \circ \text{sim}) \otimes \mathbf{r}_{\mathcal{K}}(\rho), \quad (6)$$

$$\mathbf{r}_{\mathcal{J}}((\mu \circ \text{sim}) \otimes \rho) \cong (\tilde{\mu} \circ \det) \otimes \mathbf{r}_{\mathcal{J}}(\rho). \quad (7)$$

with $\det : (\text{GL}(2, q)^2)^0 \rightarrow \mathbb{F}_q^\times$, $(a, b) \mapsto \det a$.

3 Proof of Theorem 1.1

Proposition 3.1. *For an admissible representation (σ, V_σ) of $\mathrm{GSp}(2, F)$ and a character $\mu_1 : F^\times \rightarrow \mathbb{C}^\times$, the parahoric restriction at \mathcal{J} of the Klingen induced representation $\mu_1 \rtimes \sigma$ is*

$$\mathbf{r}_{\mathcal{J}}(\mu_1 \rtimes \sigma) \cong [\widetilde{\mu}_1 \times 1, \widetilde{\sigma}] \oplus [\widetilde{\sigma}, \widetilde{\mu}_1 \times 1] \quad (8)$$

for $\widetilde{\mu}_1 = \mathbf{r}_{\mathfrak{o}^\times}(\mu_1)$ and $\widetilde{\sigma} = \mathbf{r}_{\mathrm{GL}(2, \mathfrak{o})}(\sigma)$.

Proof. An explicit model of $\mathbf{r}_{\mathcal{J}}(\mu_1 \rtimes \sigma)$ is given by the right-action of \mathcal{J} on

$$\widetilde{V} = \{f : G \rightarrow V_\sigma \mid f(pgk) = \delta_Q^{1/2}(p)(\mu_1 \boxtimes \sigma)(p)f(g) \ \forall p \in Q, g \in G, k \in \mathcal{J}^+\}.$$

By (5), any $f \in \widetilde{V}$ is uniquely determined by its restriction to \mathcal{J} and $s_1\mathcal{J}$, so the \mathcal{J} -representation \widetilde{V} is isomorphic to the direct sum

$$\{f|_{\mathcal{J}} : \mathcal{J} \rightarrow V_\sigma \mid f \in \widetilde{V}\} \oplus \{f|_{s_1\mathcal{J}} : s_1\mathcal{J} \rightarrow V_\sigma \mid f \in \widetilde{V}\}.$$

Every $f|_{\mathcal{J}}$ in the first subspace is left invariant under $\mathcal{J}^+ \cap Q$, so it maps to the σ -invariants under $\mathrm{GSp}(2, \mathfrak{o})^+$. Thus $f|_{\mathcal{J}}$ factors over a unique function

$$\widetilde{f} : \mathcal{J} / \mathcal{J}^+ \rightarrow V_\sigma^{\mathrm{GSp}(2, \mathfrak{o})^+} \quad \text{with} \quad \widetilde{f}(qg) = \widetilde{\mu}(q_{11})\widetilde{\sigma} \begin{pmatrix} q_{22} & q_{24} \\ q_{42} & q_{44} \end{pmatrix} \widetilde{f}(g)$$

for every $g \in \mathcal{J} / \mathcal{J}^+ \cong (\mathrm{GL}(2, q)^2)^0$ and every

$$q \in (\mathcal{J} \cap Q) \mathcal{J}^+ / \mathcal{J}^+ \cong ((** \ *) \times (** \ *)) \cap (\mathrm{GL}(2, q)^2)^0.$$

By definition of the isomorphism $\mathcal{J} / \mathcal{J}^+ \cong (\mathrm{GL}(2, q)^2)^0$, the space of these \widetilde{f} is the induced representation $[\widetilde{\mu} \times 1, \widetilde{\sigma}]$.

For the second subspace the argument is analogous. \square

Proposition 3.2. *For smooth characters μ_0, μ_1, μ_2 of F^\times and an irreducible admissible representation (σ, V_σ) of $\mathrm{GL}(2, F)$, the parahoric restriction of the Siegel induced representation $\sigma \rtimes \mu_0$ at the standard paramodular subgroup \mathcal{J} is*

$$\mathbf{r}_{\mathcal{J}}(\sigma \rtimes \mu_0) \cong \begin{cases} \widetilde{\mu}_0[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_2] + \widetilde{\mu}_0[1 \times \widetilde{\mu}_2, 1 \times \widetilde{\mu}_1] & \sigma \cong \mu_1 \times \mu_2, \\ \widetilde{\mu}_0[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_1] & \sigma = \mu_1 \mathrm{St}, \mu_1 \mathbf{1}, \\ 0 & \sigma \text{ cuspidal.} \end{cases}$$

Proof. By (7) we can assume without loss of generality that $\mu_0 = 1$. An explicit model \widetilde{V} of $\mathbf{r}_{\mathcal{J}}(\sigma \rtimes 1)$ is given by right-multiplication with elements of \mathcal{J} on the vector space of smooth functions $f : G \rightarrow V_\sigma$ with

$$f(pgk) = \delta_P^{1/2}(p) \cdot \sigma \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} f(g)$$

for $p \in P$, $g \in G$ and $k \in \mathcal{J}^+$. By the decomposition $G = P\mathcal{J}$ (5), every such f is uniquely determined by its restriction to \mathcal{J} . Therefore \tilde{V} is isomorphic to the vector space of \mathcal{J}^+ -invariant functions

$$\tilde{f}: \mathcal{J} \rightarrow V_\sigma$$

which satisfy the condition

$$\tilde{f}(pg) = \sigma \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \tilde{f}(g) \quad \forall g \in \mathcal{J}, \quad \forall p \in P \cap \mathcal{J}.$$

Since \tilde{f} is left-invariant under every $p \in P \cap \mathcal{J}^+$, the value $\tilde{f}(g) \in V_\sigma$ is invariant under $\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \subseteq \mathrm{GL}(2, F)$. That means $\tilde{f}(g)$ must be contained in the parahoric restriction $\mathbf{r}_{\mathcal{B}_{\mathrm{GL}(2)}}(\sigma)$ with respect to the standard Iwahori $\mathcal{B}_{\mathrm{GL}(2)} \subseteq \mathrm{GL}(2, F)$. By Lemma 2.3, $\mathbf{r}_{\mathcal{B}_{\mathrm{GL}(2)}}(\sigma)$ is zero for cuspidal σ . For $\sigma = \mu\mathbf{1}, \mu\mathrm{St}$ it is isomorphic to $\widetilde{\mu}_1 \boxtimes \widetilde{\mu}_1$ and the condition on \tilde{f} is

$$\tilde{f}(pg) = \widetilde{\mu}_1(p_{11})\widetilde{\mu}_1(p_{22})\tilde{f}(g), \quad \forall g \in \mathcal{J}/\mathcal{J}^+, \quad \forall p \in (P \cap \mathcal{J})/(P \cap \mathcal{J}^+).$$

By construction of the isomorphism $\mathcal{J}/\mathcal{J}^+ \cong (\mathrm{GL}(2, q)^2)^0$, the action of \mathcal{J} on \tilde{V} is the induced representation $[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_1]$. For the principal series $\sigma = \mu_1 \times \mu_2$, the argument is analogous. \square

Proposition 3.3. *For admissible representations of $\mathrm{GSp}(4, F)$ that are parabolically induced, the parahoric restriction at \mathcal{K} is given by*

$$\begin{aligned} \mathbf{r}_{\mathcal{K}}(\mu_1 \times \mu_2 \rtimes \mu_0) &\cong \widetilde{\mu}_1 \times \widetilde{\mu}_2 \rtimes \widetilde{\mu}_0, \\ \mathbf{r}_{\mathcal{K}}(\mu_1 \rtimes \sigma) &\cong \widetilde{\mu}_1 \rtimes \mathbf{r}_{\mathrm{GSp}(2, \mathfrak{o})}(\sigma), \\ \mathbf{r}_{\mathcal{K}}(\sigma \rtimes \mu_0) &\cong \mathbf{r}_{\mathrm{GL}(2, \mathfrak{o})}(\sigma) \rtimes \widetilde{\mu}_0, \end{aligned}$$

for smooth characters $\mu_i: F^\times \rightarrow \mathbb{C}^\times$ and an admissible representation σ of $\mathrm{GL}(2, F)$.

Proof. The proof is similar to the previous propositions using $G = B\mathcal{K}$. \square

Thm. 1.1 i). We only discuss the case of odd q ; for even q the proof is analogous. Irreducible representations ρ of type I, II, III, VII and X are parabolically induced, so the result is clear by Prop. 3.3, by (4) and Table 2. Otherwise, ρ is a non-trivial subquotient of a parabolically induced representation κ and $\mathbf{r}_{\mathcal{K}}(\kappa)$ is given by Prop. 3.3. If $\mathbf{r}_{\mathcal{K}}(\kappa) = 0$, then $\mathbf{r}_{\mathcal{K}}(\rho) = 0$ by exactness, otherwise $\mathbf{r}_{\mathcal{K}}(\rho)$ is a non-zero subquotient of $\mathbf{r}_{\mathcal{K}}(\kappa)$ by Prop. 2.2. It remains to determine the correct constituents of $\mathbf{r}_{\mathcal{K}}(\kappa)$ case by case. By (6) we can assume without loss of generality that $\mu_0 = 1$.

For the trivial representation $\rho = \mathbf{1}_{\mathrm{GSp}(4, F)}$ (type IVd), the hyperspecial parahoric restriction is trivial $\mathbf{r}_{\mathcal{K}}(\rho) = \theta_0$. By [6, (2.9)] and character theory [9]

$$\mathbf{r}_{\mathcal{K}}(L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2, F)})) + \mathbf{r}_{\mathcal{K}}(\mathbf{1}_{\mathrm{GSp}(4, F)}) \cong \mathbf{1}_{\mathrm{GL}(2, q)} \rtimes 1 = \chi_3(1, 1)$$

decomposes as $\chi_3(1, 1) = \theta_0 + \theta_1 + \theta_3$, so for type IVb

$$\mathbf{r}_{\mathcal{K}}(L(\nu^2, \nu^{-1} \text{St}_{\text{GSp}(2, F)})) = \theta_1 + \theta_3.$$

By the same argument we determine the parahoric restriction for type IVa and IVc as constituents of $\chi_1(1, 1) = 1 \rtimes \mathbf{1}_{\text{GSp}(2, q)}$ and $\chi_2(1, 1) = 1 \rtimes \text{St}_{\text{GSp}(2, q)}$.

The representation $\rho = L(\nu^{1/2} \xi \text{St}, \nu^{-1/2})$ of type Vb is a constituent of both

$$\nu^{1/2} \xi \text{St}_{\text{GL}(2, F)} \rtimes \nu^{-1/2} \quad \text{and} \quad \nu^{1/2} \xi \mathbf{1}_{\text{GL}(2, F)} \rtimes \nu^{-1/2} \xi$$

[6, (2.10)]. Therefore the parahoric restriction $\mathbf{r}_{\mathcal{K}}(\rho)$ must be a (non-zero) constituent of both $\xi \text{St}_{\text{GL}(2, q)} \rtimes 1$ and $\xi \text{St}_{\text{GL}(2, q)} \rtimes \xi$. By [9], the only common constituent is θ_1 for unramified ξ and τ_2 for tamely ramified ξ . By exactness, types Va, Vc and Vd are clear.

The Klingen induced representation $1 \rtimes \text{St}_{\text{GL}(2, F)}$ splits into the direct sum of $\tau(S, \nu^{-1/2})$ of type VIa and $\tau(T, \nu^{-1/2})$ of type VIb [6, (2.11)]. Its parahoric restriction is $\mathbf{r}_{\mathcal{K}}(1 \rtimes \text{St}_{\text{GL}(2, F)}) = \theta_1 + \theta_3 + \theta_5$. The representation $\tau(S, \nu^{-1/2})$ is contained in $\nu^{1/2} \text{St} \rtimes \nu^{-1/2}$ with restriction at \mathcal{K} given by $\text{St}_{\text{GL}(2, q)} \rtimes 1 = \theta_1 + \theta_4 + \theta_5$; while $\tau(T, \nu^{-1/2})$ is contained in $\nu^{1/2} \mathbf{1} \rtimes \nu^{-1/2}$ with restriction $\theta_0 + \theta_1 + \theta_3$. Therefore the pair of parahoric restrictions $(\mathbf{r}_{\mathcal{K}}(\tau(S, \nu^{-1/2})), \mathbf{r}_{\mathcal{K}}(\tau(T, \nu^{-1/2})))$ is either $(\theta_5 + \theta_1, \theta_3)$ or $(\theta_5, \theta_1 + \theta_3)$. But the virtual representation

$$\tau(S, \nu^{-1/2}) - \tau(T, \nu^{-1/2})$$

is the endoscopic lift of $(\text{St}_{\text{GL}(2, F)}, \text{St}_{\text{GL}(2, F)})$ in the sense of [14], so the trace of its parahoric restriction at \mathcal{K} is zero on the $\text{GSp}(4, q)$ -conjugacy class stably conjugate to $\text{diag}(a^q, a, a^{q^3}, a^{q^2})$ [7, Cor. 4.25] for $a \in \mathbb{F}_q^\times$ with $a^{q^2+1} \in \mathbb{F}_q^\times$ and $a^{q-1} \neq \pm 1$. This implies $\mathbf{r}_{\mathcal{K}}(\tau(S, \nu^{-1/2})) = \theta_5 + \theta_1$ and $\mathbf{r}_{\mathcal{K}}(\tau(T, \nu^{-1/2})) = \theta_3$. Types VIc and VId are clear by exactness.

For an irreducible cuspidal admissible representation π of $\text{GL}(2, F)$ of depth zero, the Klingen induced representation $1 \rtimes \pi$ is a direct sum of two irreducible constituents, the generic $\tau(S, \pi)$ of type VIIa and the non-generic $\tau(T, \pi)$ of type VIIb. The parahoric restriction $\mathbf{r}_{\mathcal{K}}(1 \rtimes \pi) \cong 1 \rtimes \tilde{\pi} = X_3(\Lambda, 1) = \chi_7(\Lambda) + \chi_8(\Lambda)$ has two irreducible constituents, so Prop. 2.2 implies that $\mathbf{r}_{\mathcal{K}}(\tau(S, \pi))$ is isomorphic to one of them and $\mathbf{r}_{\mathcal{K}}(\tau(T, \pi))$ is isomorphic to the other. By a suitable character twist (6) we can assume that π is unitary. Then the virtual representation $\tau(S, \pi) - \tau(T, \pi)$ is the local endoscopic character lift of the representation (π, π) of $\text{GL}(2, F)^2 / \text{GL}(1, F)$ (antidiagonally embedded), compare [14, Thm. 4.5]. For $\alpha, \beta \in \mathbb{F}_{q^2}^\times$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}_q^\times$, the trace of $\mathbf{r}_{\mathcal{K}}(\tau(S, \pi) - \tau(T, \pi))$ on the stable conjugacy class with eigenvalues $\alpha\beta, \alpha\beta^q, \alpha^q\beta, \alpha^q\beta^q$ is

$$2(\Lambda(\alpha) + \Lambda(\alpha^q))(\Lambda(\beta) + \Lambda(\beta^q)),$$

see [7, Cor. 4.20]. By Shinoda's character table [9], this coincides with the character value of $\chi_8(\Lambda) - \chi_7(\Lambda)$, but not with $\chi_7(\Lambda) - \chi_8(\Lambda)$. This implies $\mathbf{r}_{\mathcal{K}}(\tau(S, \pi)) \cong \chi_8(\Lambda)$ and $\mathbf{r}_{\mathcal{K}}(\tau(T, \pi)) \cong \chi_7(\Lambda)$.

For type IX, see [7, §3.2.1].

Let $\rho = \delta(\nu^{1/2}\pi, \nu^{-1/2})$ be an irreducible representation of type XIa where π is a cuspidal irreducible admissible representation of $\mathrm{GL}(2, F)$ with trivial central character. Then $\mathbf{r}_{\mathcal{K}}(\rho)$ must be one of the two irreducible subquotients of

$$\mathbf{r}_{\mathcal{K}}(\nu^{1/2}\pi \rtimes \nu^{-1/2}) = \tilde{\pi} \rtimes 1 = X_2(\Lambda, 1) = \chi_5(\omega_\Lambda, 1) + \chi_6(\omega_\Lambda, 1).$$

By [6, Table A.12], ρ has paramodular level ≥ 3 and therefore does not admit non-zero invariants under the second paramodular congruence subgroup, which is conjugate to

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} \cap \mathcal{K}.$$

By character theory, $\chi_5(\omega_\Lambda, 1)$ admits non-zero invariants under the image of this group in $\mathcal{K}/\mathcal{K}^+$, so $\mathbf{r}_{\mathcal{K}}(\rho)$ cannot be $\chi_5(\omega_\Lambda, 1)$. The rest follows from exactness and Prop. 2.2. \square

Thm. 1.1 ii). By transitivity of parahoric restriction (3), this is implied by Thm. 1.1. Parabolic restriction for $\mathrm{GSp}(4, q)$ can be determined explicitly by character theory. \square

For the paramodular subgroup $\mathcal{J} \subseteq \mathrm{GSp}(4, F)$, the Atkin Lehner involution provides a symmetry condition:

Lemma 3.4. *Let ρ be an irreducible admissible representation of $\mathrm{GSp}(4, F)$. The parahoric restriction $\mathbf{r}_{\mathcal{J}}(\rho)$ is isomorphic to $(a, b) \mapsto \mathbf{r}_{\mathcal{J}}(\rho)(b, a)$.*

Proof. Conjugation by u_1 preserves \mathcal{J} and \mathcal{J}^+ and gives rise to the automorphism $(a, b) \mapsto (b, a)$ of $(\mathrm{GL}(2, q)^2)^0 \cong \mathcal{J}/\mathcal{J}^+$. \square

Thm. 1.1 iii). By (7), we can assume without loss of generality that $\mu_0 = 1$. The irreducible admissible representations ρ of type I, IIIa, IIIb and VII are Klingen induced and the statement is implied by Prop. 3.1. Representations of type IIa, IIb, X, XIa and XIb are Siegel induced and given by Prop. 3.2.

For the trivial representation $\mathbf{1}_{\mathrm{GSp}(4, F)}$ of type IVd, the parahoric restriction at \mathcal{J} is clearly the trivial representation $[\mathbf{1}, \mathbf{1}]$. The representation $\rho = L(\nu^{3/2}\mathrm{St}, \nu^{-3/2})$ of type IVc is the non-trivial constituent of the Klingen induced representation $\nu^2 \times \nu^{-1} \mathbf{1}_{\mathrm{GSp}(2, F)}$ [6, (2.9)]. By exactness and Prop. 3.1, its parahoric restriction is $\mathbf{r}_{\mathcal{J}}(\rho) = [\mathrm{St}, \mathbf{1}] + [\mathbf{1}, \mathrm{St}] + [\mathbf{1}, \mathbf{1}]$. By the analogous argument with Prop. 3.2, the parahoric restriction for representations of type IVa and IVb is clear.

For $\rho = L(\nu^{1/2}\xi\mathrm{St}, \nu^{-1/2}\xi)$ of type Vc with an unramified quadratic character $\xi = \xi_u$, the parahoric restriction $\mathbf{r}_{\mathcal{J}}(\rho)$ is contained in $\mathbf{r}_{\mathcal{J}}(\nu^{1/2}\xi_u \mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}) = [1 \times 1, 1 \times 1]$.

By [7, Thm. 3.30], $\mathbf{r}_{\mathcal{J}}(\rho)$ has a generic subquotient, which must be $[\text{St}, \text{St}]$. There is exactly one further constituent in $\mathbf{r}_{\mathcal{J}}(\rho)$, because the Klingen parahoric restriction $\mathbf{r}_{\mathcal{Q}}(\rho)$ contains two constituents by Table 4. By Lemma 3.4, this can only be $[\mathbf{1}, \mathbf{1}]$. For tamely ramified quadratic character $\xi = \xi_t$, the parahoric restriction $\mathbf{r}_{\mathcal{J}}(\rho)$ is given by [7, §3.3.3]. For types Va, Vb and Vd the result is clear by Prop. 3.2 and exactness.

The representation $\rho = \tau(S, \nu^{-1/2})$ of type VIa is a constituent of the Klingen induced representation $\nu^{1/2} \text{St}_{\text{GL}(2, F)} \rtimes \nu^{-1/2}$ and of the Siegel induced representation $1 \rtimes \text{St}_{\text{GSp}(2, F)}$ [6, (2.11)]. By Prop. 3.1 and 3.2, the parahoric restriction at \mathcal{J} is a subquotient of $[\text{St}, \text{St}] + [\mathbf{1}, \text{St}] + [\text{St}, \mathbf{1}]$. The Klingen parahoric restriction $\mathbf{r}_{\mathcal{Q}}(\rho)$ has three irreducible constituents, so (3) implies $\mathbf{r}_{\mathcal{J}}(\rho) \cong [\text{St}, \text{St}] + [\mathbf{1}, \text{St}] + [\text{St}, \mathbf{1}]$. For types VIb, VIc and VId the result is clear by exactness.

Representations of type VIII are irreducible subquotients of $1 \rtimes \pi$ with π of depth zero. Their paramodular restriction is either $[\tilde{\pi}, \mathbf{1}] + [\mathbf{1}, \tilde{\pi}]$ or $[\tilde{\pi}, \text{St}] + [\text{St}, \tilde{\pi}]$ by Prop. 3.1 and Lemma 3.4. The rest of the argument is analogous to the hyperspecial case: By [7, Cor. 4.23], the character value of

$$\mathbf{r}_{\mathcal{J}}(\tau(S, \pi) - \tau(T, \pi))$$

on the conjugacy class stably conjugate to $(\text{diag}(\alpha\beta, \alpha^q\beta^q), \text{diag}(\alpha\beta^q, \alpha^q\beta))$ for $\alpha, \beta \in \mathbb{F}_{q^2}^\times$ with $\alpha, \beta, \alpha\beta, \alpha\beta^q \notin \mathbb{F}_q^\times$ is given by

$$\begin{aligned} & -2(\Lambda(\alpha) + \Lambda(\alpha^q))(\Lambda(\beta) + \Lambda(\beta^q)) \\ & = 2(-\Lambda(\alpha\beta) - \Lambda^q(\alpha\beta)) + 2(-\Lambda(\alpha\beta^q) - \Lambda^q(\alpha\beta^q)). \end{aligned}$$

This implies $\mathbf{r}_{\mathcal{J}}(\tau(T, \pi)) = [\tilde{\pi}, \text{St}] + [\text{St}, \tilde{\pi}]$ and $\mathbf{r}_{\mathcal{J}}(\tau(S, \pi)) = [\tilde{\pi}, \mathbf{1}] + [\mathbf{1}, \tilde{\pi}]$.

For type IX, see [7, §3.3.2]. □

4 Tables

The irreducible admissible representations of $\text{GSp}(4, F)$ have been classified by Sally and Tadić [8]. We use the notation of Roberts and Schmidt [6].

For $i = 0, 1, 2$ let $\mu_i : F^\times \rightarrow \mathbb{C}^\times$ be tamely ramified or unramified characters. Let π be an arbitrary cuspidal irreducible admissible representation of $\text{GL}(2, F)$ of depth zero. Its hyperspecial restriction $\mathbf{r}_{\text{GL}(2, o)}(\pi) = \tilde{\pi} = \tilde{\pi}_\Lambda$ is an irreducible cuspidal representation of $\text{GL}(2, q)$. Up to a sign it is the Deligne-Lusztig representation attached to a character Λ of $\mathbb{F}_{q^2}^\times$ in general position, i.e. $\Lambda \neq \Lambda^q$. The contragredient of $\tilde{\pi}$ is denoted $\tilde{\pi}^\vee$.

The non-trivial unramified quadratic character of F^\times is denoted ξ_u . For odd q let ξ_t be one of the two tamely ramified quadratic characters which reduce to the non-trivial quadratic character λ_0 of \mathbb{F}_q^\times . For even q there is no tamely ramified quadratic character.

Table 3. For even q , irreducible characters of $\text{Sp}(4, q)$ have been classified by Enomoto [4]. By the isomorphism $\text{Sp}(4, q) \times \text{GL}(1, q) \cong \text{GSp}(4, q)$, $(x, t) \mapsto t \cdot x$, the irreducible

characters of $\mathrm{GSp}(4, q)$ can be classified in terms of their restriction to $\mathrm{Sp}(4, q)$ and their central character. Fix a generator $\hat{\theta}$ of the cyclic character group of \mathbb{F}_q^\times and denote its restrictions to \mathbb{F}_q^\times by $\hat{\gamma}$ and to $\mathbb{F}_{q^2}^\times[q+1]$ by $\hat{\eta}$, respectively. Let $k_i \in \mathbb{Z}/(q-1)\mathbb{Z}$ be such that $\hat{\gamma}^{k_i} = \tilde{\mu}_i$. Let $l \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ be such that $\Lambda = \hat{\theta}^l$ and let l' be the image of l under the canonical projection $\mathbb{Z}/(q^2-1)\mathbb{Z} \rightarrow \mathbb{Z}/(q+1)\mathbb{Z}$ so that the restriction of Λ to $\mathbb{F}_{q^2}^\times[q+1]$ is $\hat{\eta}^{l'}$. If $(q+1)l = 0$, there is a unique preimage l'' of l under the canonical injection $\mathbb{Z}/(q+1)\mathbb{Z} \hookrightarrow \mathbb{Z}/(q^2-1)\mathbb{Z}$.

For odd q , irreducible characters of $\mathrm{GSp}(4, q)$ have been classified by Shinoda [9]. A character Λ of $\mathbb{F}_{q^2}^\times$ with $\Lambda^{q+1} = 1$ factors over a character ω_Λ of $\mathbb{F}_{q^2}^\times[q+1]$ via $\Lambda(\alpha) = \omega_\Lambda(\alpha^{q-1})$. If Λ^{q-1} is the quadratic character Λ_0 of $\mathbb{F}_{q^2}^\times$, there is a unique character λ' of $\mathbb{F}_{q^2}^\times[2(q-1)]$ with $\Lambda(\alpha) = \lambda'(\alpha^{(q+1)/2})$.

Table 4. The trivial and the Steinberg representation of $\mathrm{GL}(2, q)$ are denoted $\mathbf{1}$ and St , respectively. The parabolic induction of the character $\mu_1 \boxtimes \mu_2$ of the standard torus is denoted $\mu_1 \times \mu_2$. For typographical reasons, we write

$$\begin{aligned} A(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_0) &= \tilde{\mu}_1 \boxtimes \tilde{\mu}_2 \boxtimes \tilde{\mu}_0 + \tilde{\mu}_1 \boxtimes \tilde{\mu}_2^{-1} \boxtimes \tilde{\mu}_2 \tilde{\mu}_0 + \tilde{\mu}_1^{-1} \boxtimes \tilde{\mu}_2 \boxtimes \tilde{\mu}_1 \tilde{\mu}_0 \\ &\quad + \tilde{\mu}_1^{-1} \boxtimes \tilde{\mu}_2^{-1} \boxtimes \tilde{\mu}_1 \tilde{\mu}_2 \tilde{\mu}_0, \\ B(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_0) &= \tilde{\mu}_1 \boxtimes (\tilde{\mu}_2 \rtimes \tilde{\mu}_0) + \tilde{\mu}_1^{-1} \boxtimes (\tilde{\mu}_2 \rtimes \tilde{\mu}_1 \tilde{\mu}_0), \\ C(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_0) &= (\tilde{\mu}_1 \times \tilde{\mu}_2) \boxtimes \tilde{\mu}_0 + (\tilde{\mu}_1^{-1} \times \tilde{\mu}_2) \boxtimes \tilde{\mu}_1 \tilde{\mu}_0. \end{aligned}$$

Table 5. We call a representation of $(\mathrm{GL}(2, q)^2)^0 = \{(a, b) \in \mathrm{GL}(2, q)^2 \mid \det a = \det b\}$ *generic* if it is generic with respect to the unipotent character

$$\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right) \mapsto \tilde{\psi}(x+y).$$

This does not depend on the choice of the non-trivial additive character $\tilde{\psi}$ of \mathbb{F}_q . The irreducible representations of $\mathrm{GL}(2, q) \times \mathrm{GL}(2, q)$ are $\sigma_1 \boxtimes \sigma_2$ for irreducible representations σ_i of $\mathrm{GL}(2, q)$. We denote the restriction of $\sigma_1 \boxtimes \sigma_2$ to $(\mathrm{GL}(2, q)^2)^0$ by $[\sigma_1, \sigma_2]$. This restriction is irreducible unless $\lambda_0 \sigma_1 \cong \sigma_1$ and $\lambda_0 \sigma_2 \cong \sigma_2$ for the non-trivial quadratic character λ_0 of \mathbb{F}_q^\times , then it splits into an equidimensional direct sum of a generic constituent $[\sigma_1, \sigma_2]_+$ and a non-generic constituent $[\sigma_1, \sigma_2]_-$. These are all the irreducible representations of $(\mathrm{GL}(2, q)^2)^0$. The twist of a representation σ of $(\mathrm{GL}(2, q)^2)^0$ by a character $\tilde{\mu}$ of \mathbb{F}_q^\times is defined by

$$\tilde{\mu}[\sigma_1, \sigma_2] = [(\tilde{\mu} \circ \det) \otimes \sigma_1, \sigma_2] = [\sigma_1, (\tilde{\mu} \circ \det) \otimes \sigma_2].$$

Acknowledgements. The author wants to express his gratitude to R. Weissauer and U. Weselmann for valuable discussions.

Table 3: Parahoric restriction at \mathcal{K} for non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4, F)$.

type	ρ of $\mathrm{GSp}(4, F)$	$\mathbf{r}_{\mathcal{K}}(\rho) _{\mathrm{Sp}(4, q)}$ (even q)	$\mathbf{r}_{\mathcal{K}}(\rho)$ (odd q)	central character	$\dim \mathbf{r}_{\mathcal{K}}(\rho)$
I	$\mu_1 \times \mu_2 \rtimes \mu_0$	$\chi_1(k_1, k_2)$	$X_1(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_0)$	$\widetilde{\mu}_1 \mu_2 \widetilde{\mu}_0^2$	$(q+1)^2(q^2+1)$
IIa	$\mu_1 \mathrm{St} \rtimes \mu_0$	$\chi_{10}(k_1)$	$\chi_4(\widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1^2 \widetilde{\mu}_0^2$	$q(q+1)(q^2+1)$
IIb	$\mu_1 \mathbf{1} \rtimes \mu_0$	$\chi_6(k_1)$	$\chi_3(\widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1^2 \widetilde{\mu}_0^2$	$(q+1)(q^2+1)$
IIIa	$\mu_1 \rtimes \mu_0 \mathrm{St}$	$\chi_{11}(k_1)$	$\chi_2(\widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1 \mu_0^2$	$q(q+1)(q^2+1)$
IIIb	$\mu_1 \rtimes \mu_0 \mathbf{1}$	$\chi_7(k_1)$	$\chi_1(\widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1 \mu_0^2$	$(q+1)(q^2+1)$
IVa	$\mu_0 \mathrm{St}_{\mathrm{GSp}(4, F)}$	θ_4	$\theta_5(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	q^4
IVb	$L(\nu^2, \nu^{-1} \mu_0 \mathrm{St})$	$\theta_1 + \theta_2$	$\theta_1(\widetilde{\mu}_0) + \theta_3(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^3 + q^2 + q$
IVc	$L(\nu^{3/2} \mathrm{St}, \nu^{-3/2} \mu_0)$	$\theta_1 + \theta_3$	$\theta_1(\widetilde{\mu}_0) + \theta_4(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^3 + q^2 + q$
IVd	$\mu_0 \mathbf{1}_{\mathrm{GSp}(4, F)}$	θ_0	$\theta_0(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	1
Va	$\delta([\xi_u, \nu \xi_u], \nu^{-1/2} \mu_0)$	$\theta_3 + \theta_4$	$\theta_4(\widetilde{\mu}_0) + \theta_5(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^4 + \frac{1}{2}q(q^2+1)$
Vb	$\delta([\xi_t, \nu \xi_t], \nu^{-1/2} \mu_0)$	—	$\tau_3(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^2(q^2+1)$
Vc	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \mu_0)$	θ_1	$\theta_1(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$\frac{1}{2}q(q+1)^2$
Vd	$L(\nu^{1/2} \xi_t \mathrm{St}, \nu^{-1/2} \mu_0)$	—	$\tau_2(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q(q^2+1)$
VIa	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \xi_u \mu_0)$	θ_1	$\theta_1(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$\frac{1}{2}q(q+1)^2$
Vb	$L(\nu \xi_u, \xi_u \rtimes \nu^{-1/2} \mu_0)$	—	$\tau_2(\widetilde{\mu}_0 \lambda_0)$	$\widetilde{\mu}_0^2$	$q(q^2+1)$
Vc	$L(\nu \xi_t, \xi_t \rtimes \nu^{-1/2} \mu_0)$	$\theta_0 + \theta_2$	$\theta_0(\widetilde{\mu}_0) + \theta_3(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$1 + \frac{1}{2}q(q^2+1)$
Vd	$\tau(S, \nu^{-1/2} \mu_0)$	—	$\tau_1(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^2 + 1$
VIa	$\tau(T, \nu^{-1/2} \mu_0)$	$\theta_1 + \theta_4$	$\theta_1(\widetilde{\mu}_0) + \theta_5(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q^4 + \frac{1}{2}q(q+1)^2$
VIb	$L(\nu^{1/2} \mathrm{St}, \nu^{-1/2} \mu_0)$	θ_2	$\theta_3(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$\frac{1}{2}q(q^2+1)$
VIc	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2} \mu_0)$	θ_3	$\theta_4(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$\frac{1}{2}q(q^2+1)$
VId		$\theta_0 + \theta_1$	$\theta_0(\widetilde{\mu}_0) + \theta_1(\widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$1 + \frac{1}{2}q(q+1)^2$
VII	$\mu_1 \rtimes \pi$	$\chi_3(k_1, l')$	$X_3(\Lambda, \widetilde{\mu}_1)$	$\widetilde{\mu}_1 \cdot \Lambda _{\mathbb{F}_q^\times}$	$q^4 - 1$
VIIIa	$\tau(S, \pi)$	$\chi_{13}(l')$	$\chi_8(\Lambda)$	$\Lambda _{\mathbb{F}_q^\times}$	$q(q-1)(q^2+1)$
VIIIb	$\tau(T, \pi)$	$\chi_9(l')$	$\chi_7(\Lambda)$	$\Lambda _{\mathbb{F}_q^\times}$	$(q-1)(q^2+1)$
IXa	$\delta(\nu \xi_u, \nu^{-1/2} \pi)$	$\chi_{13}(l')$	$\chi_8(\Lambda)$	$\Lambda _{\mathbb{F}_q^\times}$	$q(q-1)(q^2+1)$
IXb	$\delta(\nu \xi_t, \nu^{-1/2} \pi)$	—	$\tau_5(\lambda')$	$\lambda_0 \cdot \Lambda _{\mathbb{F}_q^\times}$	$q^2(q^2-1)$
IXc	$L(\nu \xi_u, \nu^{-1/2} \pi)$	$\chi_9(l')$	$\chi_7(\Lambda)$	$\Lambda _{\mathbb{F}_q^\times}$	$(q-1)(q^2+1)$
IXd	$L(\nu \xi_t, \nu^{-1/2} \pi)$	—	$\tau_4(\lambda')$	$\lambda_0 \cdot \Lambda _{\mathbb{F}_q^\times}$	$q^2 - 1$
X	$\pi \rtimes \mu$	$\chi_2(l)$	$X_2(\Lambda, \widetilde{\mu}_0)$	$\widetilde{\mu}_0^2 \cdot \Lambda _{\mathbb{F}_q^\times}$	$q^4 - 1$
XIa	$\delta(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$	$\chi_{12}(l'')$	$\chi_6(\omega_\Lambda, \widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$q(q-1)(q^2+1)$
XIb	$L(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$	$\chi_8(l'')$	$\chi_5(\omega_\Lambda, \widetilde{\mu}_0)$	$\widetilde{\mu}_0^2$	$(q-1)(q^2+1)$

Table 4: Parahoric restriction at $\mathcal{B}, \mathcal{Q}, \mathcal{P}$ for non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4, F)$.

type	ρ of $\mathrm{GSp}(4, F)$	$\mathbf{r}_{\mathcal{B}}(\rho) \in \mathrm{Rep}((\mathbb{F}_q^\times)^3)$	$\mathbf{r}_{\mathcal{Q}}(\rho) \in \mathrm{Rep}(\mathbb{F}_q^\times \times \mathrm{GSp}(2, q))$	$\mathbf{r}_{\mathcal{P}}(\rho) \in \mathrm{Rep}(\mathrm{GL}(2, q) \times \mathbb{F}_q^\times)$
I	$\mu_1 \times \mu_2 \rtimes \mu_0$	$A(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_0) + A(\widetilde{\mu}_2, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$B(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_0) + B(\widetilde{\mu}_2, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$C(\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_0) + C(\widetilde{\mu}_1, \widetilde{\mu}_2^{-1}, \widetilde{\mu}_2 \widetilde{\mu}_0)$
IIa	$\mu_1 \mathrm{St} \rtimes \mu_0$	$A(\widetilde{\mu}_1, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$B(\widetilde{\mu}_1, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1 \mathrm{St} \boxtimes \widetilde{\mu}_0 + \widetilde{\mu}_1^{-1} \mathrm{St} \boxtimes \widetilde{\mu}_0 \widetilde{\mu}_1^2$ $+ (\widetilde{\mu}_1 \times \widetilde{\mu}_1^{-1}) \boxtimes \widetilde{\mu}_0 \widetilde{\mu}_1$
IIb	$\mu_1 \mathbf{1} \rtimes \mu_0$	$A(\widetilde{\mu}_1, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$B(\widetilde{\mu}_1, \widetilde{\mu}_1, \widetilde{\mu}_0)$	$\widetilde{\mu}_1 \mathbf{1} \boxtimes \widetilde{\mu}_0 + \widetilde{\mu}_1^{-1} \mathbf{1} \boxtimes \widetilde{\mu}_0 \widetilde{\mu}_1^2$ $+ (\widetilde{\mu}_1 \times \widetilde{\mu}_1^{-1}) \boxtimes \widetilde{\mu}_0 \widetilde{\mu}_1$
IIIa	$\mu_1 \rtimes \mu_0 \mathrm{St}$	$\widetilde{\mu}_1 \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0 + \widetilde{\mu}_1^{-1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0$ $+ \mathbf{1} \boxtimes \widetilde{\mu}_1 \boxtimes \widetilde{\mu}_0 + \mathbf{1} \boxtimes \widetilde{\mu}_1^{-1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0$	$\widetilde{\mu}_1 \boxtimes \widetilde{\mu}_0 \mathrm{St} + \mathbf{1} \boxtimes \widetilde{\mu}_1 \rtimes \widetilde{\mu}_0$ $+ \widetilde{\mu}_1^{-1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0 \mathrm{St}$	$C(\widetilde{\mu}_1, 1, \widetilde{\mu}_0)$
IIIb	$\mu_1 \rtimes \mu_0 \mathbf{1}$	$\widetilde{\mu}_1 \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0 + \widetilde{\mu}_1^{-1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0$ $+ \mathbf{1} \boxtimes \widetilde{\mu}_1 \boxtimes \widetilde{\mu}_0 + \mathbf{1} \boxtimes \widetilde{\mu}_1^{-1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0$	$\widetilde{\mu}_1 \boxtimes \widetilde{\mu}_0 \mathbf{1} + \mathbf{1} \boxtimes \widetilde{\mu}_1 \rtimes \widetilde{\mu}_0$ $+ \widetilde{\mu}_1^{-1} \boxtimes \widetilde{\mu}_1 \widetilde{\mu}_0 \mathbf{1}$	$C(\widetilde{\mu}_1, 1, \widetilde{\mu}_0)$
IVa	$\mu_0 \mathrm{St}_{\mathrm{GSp}(4)}$	$\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St}$	$\mathrm{St} \boxtimes \widetilde{\mu}_0$
IVb	$L(\nu^2, \nu^{-1} \mu_0 \mathrm{St})$	$3(\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0)$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1} + 2(\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St})$	$\mathrm{St} \boxtimes \widetilde{\mu}_0 + 2(\mathbf{1} \boxtimes \widetilde{\mu}_0)$
IVc	$L(\nu^{3/2} \mathrm{St}, \nu^{-3/2} \mu_0)$	$3(\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0)$	$2(\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1}) + \mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St}$	$2(\mathrm{St} \boxtimes \widetilde{\mu}_0) + \mathbf{1} \boxtimes \widetilde{\mu}_0$
IVd	$\mu_0 \mathbf{1}_{\mathrm{GSp}(4)}$	$\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1}$	$\mathbf{1} \boxtimes \widetilde{\mu}_0$
Va	$\delta([\xi, \nu \xi], \nu^{-1/2} \mu_0)$	$\widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$	$\widetilde{\xi} \boxtimes (\widetilde{\xi} \rtimes \widetilde{\mu}_0)$	$\widetilde{\xi} \mathrm{St} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \mathrm{St} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$
Vb	$L(\nu^{1/2} \xi \mathrm{St}, \nu^{-1/2} \mu_0)$	$\widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$	$\widetilde{\xi} \boxtimes (\widetilde{\xi} \rtimes \widetilde{\mu}_0)$	$\widetilde{\xi} \mathrm{St} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \mathbf{1} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$
Vc	$L(\nu^{1/2} \xi \mathrm{St}, \nu^{-1/2} \xi \mu_0)$	$\widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$	$\widetilde{\xi} \boxtimes (\widetilde{\xi} \rtimes \widetilde{\mu}_0)$	$\widetilde{\xi} \mathbf{1} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \mathrm{St} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$
Vd	$L(\nu \xi, \xi \rtimes \nu^{-1/2} \mu_0)$	$\widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \boxtimes \widetilde{\xi} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$	$\widetilde{\xi} \boxtimes (\widetilde{\xi} \rtimes \widetilde{\mu}_0)$	$\widetilde{\xi} \mathbf{1} \boxtimes \widetilde{\mu}_0 + \widetilde{\xi} \mathbf{1} \boxtimes \widetilde{\xi} \widetilde{\mu}_0$
VIa	$\tau(S, \nu^{-1/2} \mu_0)$	$3(\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0)$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1} + 2(\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St})$	$2(\mathrm{St} \boxtimes \widetilde{\mu}_0) + \mathbf{1} \boxtimes \widetilde{\mu}_0$
VIb	$\tau(T, \nu^{-1/2} \mu_0)$	$\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St}$	$\mathbf{1} \boxtimes \widetilde{\mu}_0$
VIc	$L(\nu^{1/2} \mathrm{St}, \nu^{-1/2} \mu_0)$	$\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0$	$\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1}$	$\mathrm{St} \boxtimes \widetilde{\mu}_0$
VId	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2} \mu_0)$	$3(\mathbf{1} \boxtimes \mathbf{1} \boxtimes \widetilde{\mu}_0)$	$2(\mathbf{1} \boxtimes \widetilde{\mu}_0 \mathbf{1}) + \mathbf{1} \boxtimes \widetilde{\mu}_0 \mathrm{St}$	$\mathrm{St} \boxtimes \widetilde{\mu}_0 + 2(\mathbf{1} \boxtimes \widetilde{\mu}_0)$
VII	$\mu_1 \rtimes \pi$	0	$\widetilde{\mu}_1 \boxtimes \widetilde{\pi} + \widetilde{\mu}_1^{-1} \boxtimes \widetilde{\mu}_1 \widetilde{\pi}$	0
VIIIa	$\tau(S, \pi)$	0	$\mathbf{1} \boxtimes \widetilde{\pi}$	0
VIIIb	$\tau(T, \pi)$	0	$\mathbf{1} \boxtimes \widetilde{\pi}$	0
IXa	$\delta(\nu \xi, \nu^{-1/2} \pi)$	0	$\widetilde{\xi} \boxtimes \widetilde{\pi}$	0
IXb	$L(\nu \xi, \nu^{-1/2} \pi)$	0	$\widetilde{\xi} \boxtimes \widetilde{\pi}$	0
X	$\pi \rtimes \mu_0$	0	0	$\widetilde{\pi} \boxtimes \widetilde{\mu}_0 + (\widetilde{\pi})^\vee \boxtimes \widetilde{\omega_\pi \mu_0}$
XIa	$\delta(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$	0	0	$\widetilde{\pi} \boxtimes \widetilde{\mu}_0$
XIb	$L(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$	0	0	$\widetilde{\pi} \boxtimes \widetilde{\mu}_0$

Table 5: Parahoric restriction at \mathcal{J} for non-cuspidal irreducible admissible representations of $\mathrm{GSp}(4, F)$. The index is determined by $\xi_t(\varpi) = \pm 1$.

type	ρ of $\mathrm{GSp}(4, F)$	$\mathbf{r}_{\mathcal{J}}(\rho) \in \mathrm{Rep}((\mathrm{GL}(2, q)^2)^0)$	$\dim \mathbf{r}_{\mathcal{J}}(\rho)$
I	$\mu_1 \times \mu_2 \rtimes \mu_0$	$\widetilde{\mu}_0[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_2] + \widetilde{\mu}_0[1 \times \widetilde{\mu}_2, 1 \times \widetilde{\mu}_1]$	$2(q+1)^2$
IIa	$\mu_1 \mathrm{St} \rtimes \mu_0$	$\widetilde{\mu}_0[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_1]$	$(q+1)^2$
IIb	$\mu_1 \mathbf{1} \rtimes \mu_0$	$\widetilde{\mu}_0[1 \times \widetilde{\mu}_1, 1 \times \widetilde{\mu}_1]$	$(q+1)^2$
IIIa	$\mu_1 \rtimes \mu_0 \mathrm{St}$	$\widetilde{\mu}_0[1 \times \widetilde{\mu}_1, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, 1 \times \widetilde{\mu}_1]$	$2q(q+1)$
IIIb	$\mu_1 \rtimes \mu_0 \mathbf{1}$	$\widetilde{\mu}_0[1 \times \widetilde{\mu}_1, \mathbf{1}] + \widetilde{\mu}_0[\mathbf{1}, 1 \times \widetilde{\mu}_1]$	$2(q+1)$
IVa	$\mu_0 \mathrm{St}_{\mathrm{GSp}(4, F)}$	$\widetilde{\mu}_0[\mathrm{St}, \mathrm{St}]$	q^2
IVb	$L(\nu^2, \nu^{-1} \mu_0 \mathrm{St})$	$\widetilde{\mu}_0[\mathrm{St}, \mathrm{St}] + \widetilde{\mu}_0[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}]$	$q^2 + 2q$
IVc	$L(\nu^{3/2} \mathrm{St}, \nu^{-3/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}_0[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}]$	$2q + 1$
IVd	$\mu_0 \mathbf{1}_{\mathrm{GSp}(4, F)}$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}]$	1
Va	$\delta([\xi_u, \nu \xi_u], \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}]$	$2q$
	$\delta([\xi_t, \nu \xi_t], \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[1 \times \lambda_0, 1 \times \lambda_0]_{\pm}$	$(q+1)^2/2$
Vb	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}_0[\mathrm{St}, \mathrm{St}]$	$q^2 + 1$
	$L(\nu^{1/2} \xi_t \mathrm{St}, \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[1 \times \lambda_0, 1 \times \lambda_0]_{\mp}$	$(q+1)^2/2$
Vc	$L(\nu^{1/2} \xi_u \mathrm{St}, \nu^{-1/2} \xi_u \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}_0[\mathrm{St}, \mathrm{St}]$	$q^2 + 1$
	$L(\nu^{1/2} \xi_t \mathrm{St}, \nu^{-1/2} \xi_t \mu_0)$	$\widetilde{\mu}_0[1 \times \lambda_0, 1 \times \lambda_0]_{\mp}$	$(q+1)^2/2$
Vd	$L(\nu \xi_u, \xi_u \rtimes \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}]$	$2q$
	$L(\nu \xi_t, \xi_t \rtimes \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[1 \times \lambda_0, 1 \times \lambda_0]_{\pm}$	$(q+1)^2/2$
VIa	$\tau(S, \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathrm{St}, \mathrm{St}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}] + \widetilde{\mu}_0[\mathbf{1}, \mathrm{St}]$	$q^2 + 2q$
VIb	$\tau(T, \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathrm{St}, \mathrm{St}]$	q^2
VIc	$L(\nu^{1/2} \mathrm{St}, \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}]$	1
VIId	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2} \mu_0)$	$\widetilde{\mu}_0[\mathbf{1}, \mathbf{1}] + \widetilde{\mu}_0[\mathrm{St}, \mathbf{1}] + \widetilde{\mu}_0[\mathbf{1}, \mathrm{St}]$	$2q + 1$
VII	$\mu_1 \rtimes \pi$	$[1 \times \widetilde{\mu}_1, \widetilde{\pi}] + [\widetilde{\pi}, 1 \times \widetilde{\mu}_1]$	$2(q^2 - 1)$
VIIIa	$\tau(S, \pi)$	$[\mathbf{1}, \widetilde{\pi}] + [\widetilde{\pi}, \mathbf{1}]$	$2(q - 1)$
VIIIb	$\tau(T, \pi)$	$[\mathrm{St}, \widetilde{\pi}] + [\widetilde{\pi}, \mathrm{St}]$	$2(q - 1)q$
IXa	$\delta(\nu \xi_u, \nu^{-1/2} \pi)$	$[\mathrm{St}, \widetilde{\pi}] + [\widetilde{\pi}, \mathrm{St}]$	$2(q - 1)q$
	$\delta(\nu \xi_t, \nu^{-1/2} \pi)$	$[\widetilde{\pi}, 1 \times \lambda_0]_{\mp} + [1 \times \lambda_0, \widetilde{\pi}]_{\mp}$	$q^2 - 1$
IXb	$L(\nu \xi_u, \nu^{-1/2} \pi)$	$[\mathbf{1}, \widetilde{\pi}] + [\widetilde{\pi}, \mathbf{1}]$	$2(q - 1)$
	$L(\nu \xi_t, \nu^{-1/2} \pi)$	$[\widetilde{\pi}, 1 \times \lambda_0]_{\pm} + [1 \times \lambda_0, \widetilde{\pi}]_{\pm}$	$q^2 - 1$
X	$\pi \rtimes \mu_0$		0
XIa	$\delta(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$		0
XIb	$L(\nu^{1/2} \pi, \nu^{-1/2} \mu_0)$		0

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